Appendix C
Convexity and Optimization

Here we review the basic theory of optimization problems and associated definitions.

Convex Functions and Sets

Convexity is central to the theory of optimization. We begin with a definition of a convex function:

DEFINITION C.5 *A function* $f : \Re^n \to R$ *is* **convex** *on a set* $X \subseteq \Re^n$ *if, for all* $\mathbf{x}, \mathbf{y} \in X$ *and* $\alpha \in [0,1]$

$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y})$$

If the inequality above is strict for all $\mathbf{x} \neq \mathbf{y}$, then f is said to be *strictly convex.* A function f is said to be *concave* if $-f$ is convex and *strictly concave* if $-f$ is strictly convex.

Convexity can also be defined for sets:

DEFINITION C.6 *A set* $X \subseteq \Re^n$ *is a* **convex set** *if, for all* $\mathbf{x} \in X$, $\mathbf{y} \in X$ *and* $\alpha \in [0,1]$

$$\alpha \mathbf{x} + (1-\alpha)\mathbf{y} \in X.$$

A point of the form $\alpha\mathbf{x}+(1-\alpha)\mathbf{y}$ is referred to as a *convex combination* of the points **x** and **y**. A convex set, therefore, is one with the property that any convex combination of points in the set is also contained in the set. We also have

DEFINITION C.7 *A point* **x** *is said to be an* **extreme point** *of a convex set X if there are no two distinct points* $\mathbf{y}, \mathbf{z} \in X$ *with* $\mathbf{y} \neq \mathbf{z}$ *such that* $\mathbf{x} = \alpha\mathbf{z} + (1-\alpha)\mathbf{y}$ *for some* $0 < \alpha < 1$.

In other words, **x** cannot be expressed as the convex combination of two distinct points in *X*.

Let C^1 denotes the class of continuously differentiable functions on $\Re^n$ and C^2 denote the class of all twice-continuously differentiable functions on $\Re^n$. (See below.) Here are some properties of convex functions: